THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2060B Mathematical Analysis II (Spring 2017)
HW2 Solution

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1. (P.179 Q8)

Given $\epsilon > 0$, since $\lim_{x \to a} f'(x) = A$, there exists $\delta > 0$ such that for all $y \in (a, b)$ such that $a < y < a + \delta$, $|f'(y) - A| < \epsilon$. We claim that the same δ works for the definition of differentiability of f at a: given any $x \in (a, b)$ such that $a < x < a + \delta$, since f is continuous on $[a, x]$ and differentiable on (a, x), by Mean Value Theorem (Theorem 6.2.4), there exists $y \in (a, x)$ such that

$$\frac{f(x) - f(a)}{x - a} = f'(y)$$

Since $y \in (a, x)$, $a < y < a + \delta$ and hence

$$|\frac{f(x) - f(a)}{x - a} - A| = |f'(y) - A| < \epsilon$$

Therefore, for all $\epsilon > 0$, there exists $\delta > 0$ such that for all $x \in (a, b)$ with $a < x < a + \delta$,

$$|\frac{f(x) - f(a)}{x - a} - A| < \epsilon$$

Hence, f is differentiable at a with $f'(a) = A$.

Remark: Many students argued that y tends to a as x tends to a, so $\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} f'(y) = A$. This is reasonable, but is not vigorous enough, since the notion of "tends to" can be made precise by using $\epsilon - \delta$ argument. Also, the latter equality is not immediate from assumption, as y is not a "free variable" since y depends on x (and not necessarily continuously). It's better to use the definition of limit as demonstrated above.

2. (P.179 Q11)

We will consider the function given in Section 6.1 Q10 in HW1:

$$g(x) = \begin{cases} x^2 \sin \frac{1}{x^2} & , x \neq 0 \\ 0 & , x = 0 \end{cases}$$

As shown in the solution of HW1, g is differentiable on $\mathbb{R}$. We claim that g satisfies all the requirements of this question:

(i) g is uniformly continuous on $[0, 1]$: since g is differentiable on $\mathbb{R}$, by Theorem 6.1.2, g is continuous on $\mathbb{R}$, in particular on $[0, 1]$. Therefore, by Uniform continuity theorem (Theorem 5.4.3), g is uniformly continuous on [0,1].

(ii) g is differentiable on $(0,1)$: this follows immediately from the fact that g is differentiable on $\mathbb{R}$.

(iii) g' is unbounded on $(0,1)$: this is demonstrated in the proof of unboundedness of g' on [-1,1] in HW1.

Therefore, g is a function satisfying all the requirements of this question.

3. (P.179 Q15)

Since f' is bounded on I, there exists $M \in \mathbb{R}$ such that for all $w \in I$, $|f'(w)| \leq M$.

To show f satisfies a Lipschitz condition on I, it suffices to show that there exists $L \in \mathbb{R}$ such that for all $x, y \in I$, $|f(x) - f(y)| \leq L|x - y|$

We choose $L = M$ and claim that the above statement holds true: Given any $x, y \in I$,

Case 1: $x = y$: then $|f(x) - f(y)| = 0 \leq 0 = L|x - y|$

Case 2: $x < y$: Since I is an interval, $[x, y] \subseteq I$. Since f is differentiable on I, f is differentiable on $[x, y]$, and by Theorem 6.1.2 f is continuous on $[x, y]$; also f is differentiable on (x, y). Therefore, by Mean Value Theorem (Theorem 6.2.4), there exists $c \in (x, y)$ such that

$$\frac{f(y) - f(x)}{y - x} = f'(c)$$

Hence, $|f(y) - f(x)| = |f'(c)||y - x| \leq M|y - x|$.

Case 3: $x > y$: interchanging the roles of x and y and adopt similar argument as in case 2 (i.e. replacing $[x, y]$ by $[y, x]$, etc.) , we have

$$|f(x) - f(y)| \leq M|x - y|$$

Therefore, for all $x, y \in I$, $|f(x) - f(y)| \leq L|x - y|$, and hence f satisfies a Lipschitz condition on I.

Remark: Most students overlooked the case $x = y$. Although the argument is trivial, it is still essential as this is the only case where Mean Value Theorem is not applicable; also, some students combined case 2 and 3 together by saying "...there exists c between x and y...". This is ambiguous as it is not clear whether c could possibly be x or y by saying so (in other words, whether the "between" is inclusive and exclusive). It is better to split into cases for the sake of clarity.